

# Mathematical Modeling

Applications with GeoGebra™

**Jonas Hall**  
**Thomas Lingfjärd**



**WILEY**



# **MATHEMATICAL MODELING**



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**JONAS HALL AND THOMAS LINGEFJÄRD**

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Published by John Wiley & Sons, Inc., Hoboken, New Jersey  
Published simultaneously in Canada

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### ***Library of Congress Cataloging-in-Publication Data***

Names: Hall, Jonas, 1963– | Lingefjärd, Thomas, 1952–

Title: Mathematical modeling : applications with GeoGebra / Jonas Hall, Thomas Lingefjärd.

Description: Hoboken, New Jersey : John Wiley & Sons, Inc., [2017] | Includes index.

Identifiers: LCCN 2016009974 (print) | LCCN 2016019552 (ebook) | ISBN 9781119102724 (cloth) |

ISBN 9781119102694 (pdf) | ISBN 9781119102847 (epub)

Subjects: LCSH: Mathematical analysis—Data processing. | Mathematical models—Data processing.

Classification: LCC QA300 .H353 2017 (print) | LCC QA300 (ebook) | DDC 003—dc23

LC record available at <https://lcn.loc.gov/2016009974>

Set in 10/12pt Times by SPi Global, Pondicherry, India

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

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# PREFACE

Welcome to the world of GeoGebra! GeoGebra is a mathematical environment that will allow you to work with graphing, dynamic 2D and 3D geometry, dynamic and symbolic algebra, spreadsheets, probability, complex numbers, differential equations, dynamic text, fitting functions of any kind to data, and so forth. All representations of mathematical objects are linked and allow you to view, experiment with, and analyze problems and situations in a laboratory-like setting. You can build a geometrical model in one window, animate it, and collect data to a spreadsheet or directly to a graph. GeoGebra works on computers, tablets, and iPhones, under multiple operating systems and is free to use for noncommercial purposes. It can be used in over 65 different languages delivered from a menu option.

Welcome also to the world of mathematical modeling in high school (grades 10–12). With computer tools such as GeoGebra and Wolfram Alpha you can now expand the relatively modest modeling normally done in class to include more real-life situations and solve more interesting and difficult problems. While the computer does the necessary calculations and graphing, with GeoGebra your students will learn the process of translating problem situations to the mathematical language that the computer needs to be able to work efficiently. They will also learn to interpret the results that the computer gives them and draw sensible conclusions from them.

These competencies—translating problem situations to mathematical language suitable for computer processing and analyzing, reasoning about and presenting results—are more important in a modern world than performing specific algorithmic calculations. After all, no one today complains about the fact that we no longer teach the manual algorithm for calculating square roots.

This book came about because the authors strongly believe that modeling lies at the heart of mathematics. And in order to model interesting problems, you need

powerful tools. The strong user community and fast development of GeoGebra made it the tool of choice. With it, we were able to explore a multitude of modeling situations, some quite simple and some involving systems of differential equations, something not normally taught in high school but rendered possible with the use of computers.

In Sweden, modeling is one of seven competencies the students are to develop in mathematics education. On the cover of this book, we have symbolically placed the modeling competence at the center of the other competencies, indicating our belief that modeling is absolutely central in mathematics.

In this English translation of the book, we have restructured our material so that it now comes in order of mathematical content. Thus we explore linear models, non-linear models, models requiring calculus and differential equations, and discrete and geometrical models. It is then up to you to decide what problems in this book might be suitable for what courses and what students you teach in your school.

This book should be useful for both experienced teachers and for those students who are in teacher education programs. In addition, students of mathematical modeling at starting level university courses may find this book useful, as may anyone wishing to learn GeoGebra really well. Indeed, the book is intended to be a handy reference on both modeling and GeoGebra, for the reader to keep and return to look up the details of problem solving, modeling, and GeoGebra techniques. If you are new to GeoGebra, we suggest that you read Appendix A, *An Introduction to GeoGebra*, first before continuing with the rest of the book.

Several digital resources are available with this book. On the book's website you will find a collection of all the GeoGebra files used in the book's problems, a list of clickable links, and some screencasts showing basic techniques.

Last we wish to acknowledge the continued support we have received from the GeoGebra community, which has encouraged us to write this book, and we wish to thank our families for being, above all else, patient.

*Sweden, May 2016*

JONAS HALL AND  
THOMAS LINGEJÄRD

# INTRODUCTION

## ABOUT THIS BOOK

This book is written primarily for teachers of mathematical modeling in upper secondary schools or in high schools. Students in a teacher training program at a university or studying mathematical modeling in an introductory course at the university may also want to explore the possibilities that GeoGebra can afford. The book was conceived from the standpoint of the Swedish curriculum, which regards mathematical modeling competence to be one of seven competencies that should be taught and assessed in upper secondary school.

As a school subject, mathematics is no longer only about calculation. Some parts of mathematics, of course, relate strongly to procedures and counting, but altogether this part of the curriculum has less emphasis today than it used to have. Today, mathematics is treated as a tool, as an aid, as a language, and as logic. The curriculum in many countries is nowadays expressed in terms of competency objectives. The competencies are general and not related to a specific mathematical content. Yet, the competencies are developed in levels by students' processing specific content. The modeling competency is one of these competencies that draw heavily on functions and differential equations.

Mathematical models and other mathematical representations such as diagrams, histograms, functions, graphs, tables, and symbols normally make it easier for abstract mathematical concepts to be understood and for other phenomena to be described in mathematical terms. Educators today are facing a world that is shaped by increasingly complex, dynamic, and powerful systems of information that are met through various media. Being able to interpret, understand, and work with mathematical models and other complex systems involves important mathematical

processes that become discernible and obvious when teaching mathematical modeling.

In mathematics education, as seen from the K–12 perspective, teachers work with different representations in order to help students understand mathematical objects and concepts. Models such as geometrical constructions, graphs of functions, and a variety of diagrams are used to introduce new concepts and to show relationships, dependency, and change. Mathematical models, structures, and constructions are also used in different scientific fields, such as in physics and the social sciences. To be able to construct, interpret, and understand mathematical models is becoming increasingly important for students all over the world.

Our main academic position is that once modeling competency is acquired in the classroom, all other competencies will be addressed automatically. With training in mathematical modeling, instead of always asking “Why are we doing this?” students will find classroom work to be interesting and related to reality, and then concepts, procedures, problem solving, reasoning, communication, and relevance will follow without much effort. If you, the teacher, try to do it the other way around, you may soon discover that in sticking with too many routine calculations you will end up without time to address the modeling and reasoning competencies.

There were some basic considerations that we needed to address in writing this text on mathematical modeling. We could have chosen to only focus on the process of constructing and developing models or instead on the evaluation of already produced mathematical models. We decided to try and address both situations in this book. However, for those of you teaching mathematical modeling in upper secondary school, it may be a good idea to start with existing and well-developed models. Then, as students become familiar with the mathematical modeling concept, they could be started on constructing their own mathematical models.

To place mathematical modeling into a particular branch of mathematics, one could consider it as applied problem solving using data that have already been gathered in some way. We try to address the many different data that can be used in our selection of modeling examples in order to show how mathematical models are applied everywhere in our society. In some instances, however, we investigate purely geometrical models.

In today’s schools, teachers have the possibility to allow every student to use powerful mathematical instruments that help them learn and do mathematics in a way that humans once only could dream about. Students can tackle difficult problems a lot earlier with these tools, so they can connect concepts and procedures to more realistic situations and open up their minds to a more nuanced communications.

In this book we decided to mainly work with GeoGebra, but other tools, primarily Wolfram Alpha, can be used as well. GeoGebra was created in 2001 by Marcus Hohenwarter, and as a tool, it could be considered a mathematical laboratory, or even an environment. GeoGebra is free and platform independent, and it handles algebra, plane geometry, 3D geometry, functions, statistics, spreadsheet calculations, and symbolic algebra. GeoGebra has been translated to over 50 languages and is used all over the world. In this book we show how to use GeoGebra for mathematical modeling as well as how to apply it to teaching mathematics in general.



We have organized the mathematical modeling examples in the following order:

Chapter 1: Some Introductory Problems

Chapter 2: Linear Models

Chapter 3: Nonlinear Empirical Models I

Chapter 4: Nonlinear Empirical Models II

Chapter 5: Modeling with Calculus

Chapter 6: Using Differential Equations

Chapter 7: Geometrical Models

Chapter 8: Discrete Models

Then we have added four more chapters on the teaching and assessing of mathematical modeling, in accord with the methodology of the teaching profession:

Chapter 9: Modeling in the Classroom

Chapter 10: Assessing Modeling

Chapter 11: Assessing Models

Chapter 12: Interpreting Models

For those of you who are new to GeoGebra, we have added an introduction to this interactive, dynamic platform. We have further added a function library that can be browsed for different functions to fit data.

Appendix A: Introduction to GeoGebra

Appendix B: Function Library

In trying to model different phenomena, you will soon discover that you need different prerequisites in mathematics. We address this issue with different mathematical modeling examples at different levels of learning. In this regard the mathematical hierarchy we present is probably much the same in your school system as it is around the world. If you have previous experience with GeoGebra, we recommend that you study the chapters in sequence. If you have no previous experience at all, we suggest that you study Appendix A first, and thereafter the chapters in order.

**Chapter 1–8** contain the modeling tasks. In each of these chapters there are a number of solved modeling tasks with at least one, sometimes several, thorough solution suggestions. The solutions are very detailed, both mathematically and technically. By reading—and doing—these solutions, your students will learn mathematics, mathematical modeling, and GeoGebra techniques, and so become good modelers and problem solvers. Each chapter also has a number of unsolved tasks at the end.

Each task may be varied in a number of ways. Sometimes it may be that only a value should be calculated, sometimes several different models could be created, sometimes an error estimate could be included, and sometimes it may be better to write a report. These different ways to work out a solution can be applied to all tasks.

We have varied the tasks somewhat randomly and encourage you to adapt them to your students' needs and current levels.

The modeling tasks, both those we have solved and those left to be solved are often quite comprehensive and cannot be fitted in just one lesson. We think of them as requiring students to have at least a few lessons before starting the assignment or a week to work on the assignment. You, the teacher, may choose to do nothing else for a week or so, but we prefer to think about the tasks as parallel assignments that allow the student to work on them for long time periods while learning new concepts, asking questions, discussing the tasks with classmates, and so forth. This way they will learn and progress in reasoning, communication, conceptual growth, and more.

We also believe that students need to see many examples of written mathematical reasoning in order for them to be able to start producing written mathematical reports. The typical syllabus mentions detailed and nuanced reasoning, and this competency needs to be discussed so that students know what is expected of them. It is our hope that the many solved examples in this book will be one such source of wisdom for students and serve as an inspiration for teachers to help students develop their competencies.

**Chapter 9** addresses how to organize everyday work in the classroom, and it gives some examples of different approaches to that task. Teachers can easily turn into coaches when introducing and maintaining modeling processes, so they do need to assume that students have certain inner motivation and experience together with a desire to learn. The learning that takes place is built on the assumption that the students get continual opportunities to test, validate, and rebuild their previous knowledge. Moreover in the classroom students will learn from each other, so their social interactions are important. Lecturing should be seen as a compliment, and not as the main core of teaching in the classroom.

**Chapters 10 and 11** are about assessing and evaluation. **Chapter 10** is on evaluating students' work, and **Chapter 11** is more technical and discusses how to evaluate models and do basic mathematical error analysis.

It is obviously a challenging task to evaluate mathematical models as well as mathematical modeling. In play here is not only the pure mathematical ability to analyze and select the "best" model. Anyone who does mathematical modeling also needs to attain a certain level of technical knowledge in order to do the modeling. Students who have gone through the modeling process must be given time and space to explain and discuss their work. As they fulfill the discussion and presentation requirements, their competencies will become quite apparent.

In **Chapter 10** we also describe what forms the students' mathematical modeling presentations might take.

In **Chapter 11** we look at different ways to evaluate models. Even a model that follows Newton's cooling law describing the cooling of coffee in a cup or a linear model for pole vault results can be expected to have some pervasive errors. That means that mathematical models seldom are correct in the same sense as  $2+2=4$  is correct.

When you compare a model's conditions with reality, you might find good conformity. The populations of many animals and plant species, both globally and in different countries, are decreasing exponentially. These species will, if nothing is done, go extinct. But reality is always more complex than the model. Many species

can be saved. One such example is the Swedish peregrine, which decreased in number from the 1950s to the 1970s when large rescue efforts were made. That work changed the decreasing trend of the population, and now the previous model no longer works.

You may find some populations that are growing exponentially, however. Bacteria may, under certain conditions, grow exponentially. This could also hold true for larger species such as birds. The cormorant has increased exponentially since the mid-1950s. In other situations species numbers may oscillate, often due to deep and complex feedback loops. In **Chapter 11** we describe and illustrate some basic techniques for the evaluation and estimation of the errors inherent in all models.

**Chapter 12** is about the different ways a student may interpret a modeling situation. You will see that there is a difference between interpreting the situation as an *event* or as a *process*.

In the **Appendixes** are technical descriptions of GeoGebra and math functions. **Appendix A** provides a basic introduction to GeoGebra for teachers not yet familiar with this excellent mathematical laboratory. **Appendix B** provides a function library showing how different, in upper secondary school more or less common, functions can be parameterized in order to be used in the mathematical modeling process.

This is not a textbook in mathematics. Our intention in writing this book was to provide an introduction to the subject of mathematical modeling, and not a comprehensive text on mathematical modeling. The book does present the many different ways that mathematics can be used and applied in mathematical modeling. We are convinced that anyone who learns mathematical modeling can also learn to use mathematics in a new way and therefore learn more mathematics. Through mathematical modeling we humans also learn to see the importance of mathematics in the world around us.

When should students be taught mathematical modeling during their school years? In certain countries mathematical modeling is already compulsory in primary schools, while in other countries students are taught modeling in upper secondary schools. Today, it appears that with the increasing availability of computers, teachers in many countries are becoming more and more interested in teaching mathematical modeling early in the educational system. Whatever the case is in your country, many students already think in terms of informal models when they compare different cell-phone plans. Even though the modeling exercises in this book relate to the upper secondary school level, it is up to the teacher using this book to introduce mathematical modeling when it suits the course's content.

As already mentioned, the main purpose of this book is to teach mathematical modeling but there are also other purposes:

- To serve as a guidebook for mathematical modeling with GeoGebra and thereby also as training material for mathematics teachers
- Together with additional material on the Internet (GeoGebra constructions and screencasts), to serve as a problem sets repository for teachers to use in working with mathematical modeling
- To provide solutions in such full detail as to assist students and teachers in preparing qualitative short mathematical reports

For the many teachers who have not yet worked either with mathematical models or with GeoGebra in this way before, the book may prove to be appropriate as course literature in teacher training programs or for theme-based teacher in-service training.

## ABOUT MATHEMATICAL MODELING

The main strength of mathematical modeling is learning to make decisions. With the help of mathematical modeling, sometimes predict future trends can be predicted, decisions made on global environmental issues, and even the kind of resistor to select when building a hobby radio. Mathematical modeling can be used in artwork, in creating perspectives and in compositions. Mathematical modeling is used in packaging industry, in economics, in biological systems, in medical trials, and in computational physics. It is in fact difficult to find any area where humans work that has no use at all of mathematical models.

In today's competition-driven markets, the modern engineer needs to reduce start-up times and costly trial series productions and other construction costs. Thus, virtually all product development is now dependent on successful modeling and simulation, whether we are talking about cars, cellphones, computer parts, medical equipment, or more "invisible" things such as efficient queueing strategies for airports and trains or software development.

Obviously, the computer has given mathematical modeling a huge boost. Everybody with a modern mobile telephone and access to the Internet also has access to tools like Wolfram Alpha. Science is even changing with technology. Once we talked about theoretical and experimental physics. Today, we also talk about computational physics built on powerful computational models that we could not construct before the computer arrived. Once we talked about the dividing of the world in matter and energy. Today, we also talk about the importance of information for the structuring of matter and energy. Today, we also talk about artificial intelligence and the singularity date when computers become more intelligent than humankind. Mathematical modeling is an important part of all the information we need to handle.

Here are some further arguments as to why it is important to construct models:

- The real system is impossible to experiment with. It could be a living human being or a distant stellar cluster.
- The real system is too expensive to toy around with. It could be a space exploring satellite or the regulating system of a chemical production plant.
- The real system is too dangerous to experiment with. It could be a new aircraft model or finding the correct dose of medicine.
- Modeling gets you a better understanding of the original problem.
- Modeling makes visible a system that is not yet built or constructed.
- Modeling can help you foresee the future and make prognoses.



**FIGURE 0.1** Example of a model presenting Ohms law as a (near) proportionality.

Students who enter upper secondary schools around the world have, of course, already met some simple mathematical models. One such example is Ohm’s law from compulsory school, as shown in Figure 0.1.

Ohm’s law states that the current through a conductor between two points is directly proportional to the potential difference across the two points. This introduces the constant of proportionality, the resistance, and the usual mathematical equation that describes this relationship  $U=R \cdot I$  is a proportionality between the variables  $I$ =current (Ampere),  $U$ =voltage (volt) and the parameter  $R$ =resistance (ohm).

When describing the mathematical modeling process in theoretical terms, researchers often talk about a cycle that humans go through when they are involved in the process of mathematical modeling. It could be understood in the following way:

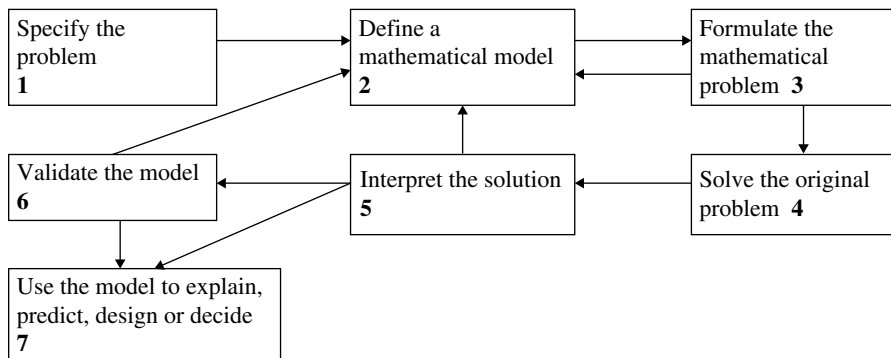
The first step in a mathematical modeling process is to translate the situation or the phenomenon at hand into a mathematical problem using mathematical terms. This means that you sometimes need to define variables or know something about the relations that exist between the variables in the example above. The original problem will go through a mathematical process, an analysis, and then, perhaps it is solved. The results should be validated against theories or common knowledge and then perhaps tested in order to see if the solution works. Figure 0.2 shows one interpretation of this process.

An example that we think makes this mathematical modeling process more understandable is the *bear problem*, a generic problem that could be used in many mathematics courses around the world.

The number of bears is increasing (in some region). In 2000 there were 2,500 bears and in 2010 there were 2,700 bears.

a) How many bears will there be in 2025?

b) When will there be 3,500 bears?



**FIGURE 0.2** Main stages in modeling (from Mason, 1988, p. 209).

In our view, this is a good task to start with, but it has to be very detailed when presented in the classroom. It contains mathematical modeling, linear functions, and exponential functions; students will learn how to find the model's or the function's parameters and how to solve different types of equations. We will illustrate step by step how the *bear problem* can be solved by hand or with GeoGebra.

### Step 1—Select and Organize a Model

To begin, you need to explain what kind of mathematical model you want the class to work with. If the text says: “The number of bears is increasing ...

- ... equally much/the same amount/by 20 bears/by the same number/linearly ... every year (month, second, hour, ...), then you should use a **linear** model:

The number of bears  $B(t) = a \cdot t + b$ .

- ... by the same rate/percentage/by 1.5 %/exponentially ... every year (month, second, hour, ...), then you should use an **exponential** model:

The number of bears  $B(t) = C \cdot r^t$ .

### Step 2—Define Your Variables

Clearly explain to the class what the variables stands for:

- $t$  is the time in years starting at year 2000 (give the **unit** and what  $t=0$  corresponds to).
- $B$  is the number of bears (or perhaps the number of **thousand** bears).

### Step 3—Define the Parameters

If you decided to put the time as  $t=0$  equal to year 2000, then the intersection between the graph and the  $y$ -axis ( $b$  or  $C$ ) will correspond to the first measurement. In this case you can directly write that: “I see that ...

- $b = 2,500$  (**linear** model), or
- $C = 2,500$  (**exponential** model).

The second parameter requires a calculation:

- In the **linear** model you must calculate  $a$  (the slope), which you do like this:

$$a = \frac{\Delta B}{\Delta t} = \frac{(2700 - 2500) \text{ bears}}{10 \text{ years}} = 20 \text{ bears / year}$$

or you could solve the equation  $B = a \cdot t + b$  and plug in  $2,700 = a \cdot 10 + 2,500$ . Note that this equation is about finding the **second measurement**. You used the **first** equation to find the value of  $m$ . The annual increase is 20 bears per year.

- Working with the exponential model in a similar way, you can solve an equation for the **second measurement**:  $B = C \cdot r^t$ , yielding  $2,700 = 2,500 \cdot r^{10}$ . Dividing both sides of the equation with 2,500, you get  $1.08 = r^{10}$ ; leading to  $r = 1.08^{1/10} = 1.0077258$ , which means an annual increase of about 0.77%.

In GeoGebra the points can be defined as **A** = (0, 2500) and **B** = (10, 2700). You get a linear model with the command **FitPoly[A, B, 1]** or an exponential model with **FitGrowth[A, B]**. You can find the value of the parameters from the algebra window.

#### Step 4—Write up Your Conclusions by Formulating Your Model

Write: “I have found that the number of bears can be predicted by ...

- $B(t) = 20 \cdot t + 2500$  (**linear** model),” or
- $B(t) = 2500 \cdot 1.0077258^t$  (**exponential** model).”


#### Step 5—Find the Number of Bears after a Certain Time

Since you now have the model, all you have to do is to enter the value of the time into your model. Be careful when you explain how you use the  $t$ -value. The year 2025 corresponds to  $t = 25$  and so:

- $B(t) = 20 \cdot 25 + 2500 = 3000$  bears (**linear** model), or
- $B(t) = 2500 \cdot 1.0077258^{25} = 3030$  bears (**exponential** model).

In GeoGebra you type

$$x = 25$$

and can then decide where this line intersects with the graph by asking GeoGebra to construct an intersection with the Intersect tool .

### Step 6—Find When the Bears Come to a Certain Number

In a similar way you can solve equations:

- $B(t) = 20 \cdot t + 2500$  with  $B = 3500$ , yielding the equation  $3500 = 20 \cdot t + 2500$ , which gives  $t = 1000/20 = 50$  years corresponding to the year 2050 (**linear model**), or
- $B(t) = 2500 \cdot 1.0077258^t$  with  $B = 3500$ , yielding the equation  $3500 = 2500 \cdot 1.0077258^t$ , where  $1.4 = 1.0077258^t$  and  $t = (\lg 1.4) / (\lg 1.0077258) = 43.7$  years, corresponding to the year 2043 (**exponential model**).

In GeoGebra you type

$$y = 3500$$

and find the intersection point as before.

As you notice, all of the steps in the modeling cycle are not always present in all mathematical modeling tasks and all the practical steps do not always correlate one-to-one with the theoretical steps. However, as we will show further on in the book, when we get to more complex modeling exercises, we will often walk around the modeling cycle a few times

We will round up this excursion into practicality by looking at another phenomenon that actually affects our life regardless of where we live or of what we do. This phenomenon is usually called **price elasticity**. The basic question is the following:

*How much will the demand for a product decrease if the price is increased?*

We begin by making a basic assumption that there is a proportionality between the relative changes in price and demand. This is a reasonable assumption, at least when we are talking about small changes. Using algebra this situation can be defined as

$$\frac{q - q_0}{q_0} = \varepsilon \frac{p - p_0}{p_0}$$

where  $p$  is the price,  $q$  is the quantity of the product, and  $\varepsilon$  is the price elasticity. We can write  $q$  as a function of  $p$  and get  $q = q(p)$ , where  $p$  is the price  $q$  is the quantity of the product. If we only have two observations of price and quantity for a product,  $(p_1, q_1)$  and  $(p_2, q_2)$ , then the average price elasticity can be defined as

$$\varepsilon = \frac{q_1 - q_2}{p_1 - p_2} \cdot \frac{p_1 + p_2}{q_1 + q_2}$$

So, in order to understand a model of price elasticity—the price that consumer are willing to pay for a product—you have to learn and understand quite a lot of mathematics such as rational expressions, linear approximation, proportionality, and negative numbers. In fact you will almost always end up in fairly complicated models by just asking apparently easy questions. Here are some examples:



- What happens in a lake when the temperature changes? How long a time does it take for the lake to turn into ice? How long will it take for the iced lake to turn into water again?
- Think of two or more intersecting roads. Should we have a traffic light system or a rotary? What would a model look like?
- How fast will a forest fire spread?
- How fast will a contaminated lake clean up?
- In games on computers—how will objects move? In what way do we want three-dimensional objects to appear in a two-dimensional monitor?
- In heating a house, it is well known that countries in the upper Northern Hemisphere have houses constructed with a short north wall containing few windows. But how short is the wall and how many windows are usually present?
- In our connected world, the placement of telecommunication masts is important. The economic principle is to cover as large area as possible with every mast. The result, when analyzing this phenomenon, will be a geometrical model. We will also have a choice of geometrical objects, such as regular hexagons or circles.

The types of questions asked are often categorized in four groups: understanding the system, predicting the system, regulating the system, and constructing the system. Understanding is the most basic category. Prediction is not possible without understanding, regulation is not possible without predictive power, and constructing something requires you to be able to control and regulate its behavior.

In teaching mathematical modeling, it is crucial that you get the students used to the different models and techniques. They should in fact go several times through the steps or stages in the modeling cycle.

Within different academic fields there is a need to study the relationships between measured variables. When you are constructing a mathematical model that describes the relationships between the variables mathematically, you are giving yourself the opportunity to predict and explain what will happen in the future. Therefore you, the teacher, should ensure that your students accomplish the following by working with mathematical modeling:

- Learn to problematize realistic situations
- Learn to fulfill assignments
- Learn about a variety of subject areas where people work with mathematics
- Learn to write mathematical reports with correct mathematical language
- Learn to use powerful tools and methods
- Learn to use powerful digital tools as GeoGebra and Wolfram Alpha
- Achieve different mathematical competencies from practicing mathematical modeling

All the different mathematical modeling tasks that we discuss in this book are solved for you, but the solutions are merely suggestions. Several of the problems can be solved in other ways.

## ABOUT MATHEMATICAL MODELING TECHNIQUES

Among the different basic techniques within mathematical modeling, perhaps the most basic is to just fit a functional dependency to a given data set. That technique is sometimes called *empirical mathematical modeling*, and it is useful for *interpolation* (i.e., used to predict values *within* the range) but less useful for *extrapolating* (i.e. used to predict result *outside* the range). If there is an underlying theory, that can help make extrapolations work better.

Theoretical considerations make it possible to create mathematical models independent of measured data. Such a model often contains a few parameters that can be adjusted to get the best fit for the measurements you have done. One such common example is that coffee cools off exponentially down to room temperature, which gives you a model of the form  $T = T_0 \cdot a^t + T_R$ , where  $T_0$  is the temperature difference against the room temperature at the time  $t=0$ ,  $T_R$  is the room temperature, which you probably know, and  $a$  is a constant that directs the speed of the cooling. The parameters  $T_0$ ,  $a$ , and eventually  $T_R$  can be varied to get the model to adjust as well as possible to the measured values. The process is often called “regression analysis” and GeoGebra have very strong tools for doing this.

If you do not have access to your own data, you may get data in the graph window by building representations of physical models that you can investigate and measure. These measurements give data that you can analyze in another window, as in Problem 8.6 *Inner Areas in a Triangle*. This is what is known as doing *simulations*. A specific sort of simulation is when you are working with pure mathematical objects, such as triangles and squares. In our case, we don’t normally say that we are doing a simulation, but the technique is the same.

Another sort of simulations is called *stochastic*, or *Monte Carlo simulations*. These simulations are built on the concept of randomizing the control of certain variables. By doing the simulation many times, we can investigate a model in another way than a theoretical model would be able to do. It is possible to find quite many simulations on the Internet. Two useful websites with such collections are <http://www.shodor.org/interactivate/activities/> and <http://dmentrard.free.fr/geogebra/index.htm>.

Another sort of modeling, sometimes called *deterministic* in contrast to *stochastic*, is based on the creation of *differential equations* that you can solve and adapt to start values. In chapter 6 we will learn the powerful ways to solve complex problems. Without today’s technology you would probably only be able to solve the simplest different equations, but with the aid of GeoGebra and Wolfram Alpha you have the capability to solve very advanced models and with solutions that you can find symbolically or numerically.

In some types of problems you will have systems of differential equations, as when studying how the numbers of predators and prey affect each other’s living conditions. You can then do more analyzing with the help of phase diagram, where the number of predators set on one axis and the number of prey on the other axis.

There are, of course, other ways to do mathematical modeling, but in this book we have chosen the techniques described above because we believe these to be quite suitable for upper secondary schools.

## ABOUT BEING A GOOD MODELER

What does it take to become a good mathematical modeler and how do you orchestrate your teaching so that your students become good at mathematical modeling? The students need to develop certain skills:

- Competence enough to reason about the real-life situation and thereby to connect to it and build a mathematical model that relates to reality. Students could, for instance, understand the relationship between the price of an ice-cream cone and the number of ice cream scoops you get, or for the higher level student, that the upper bridge span on Sydney Harbour bridge can be modeled by  $y = 1/(x^2 + 1)$ .
- Experience using different basic functions and how these functions are applied and behave. In particular, what straight lines look like, odd or even polynomials, rational functions, exponential functions, trigonometric functions, logistical and hyperbolic functions, the normal distribution function, and then again, the products of functions.
- Understanding as to how to move and stretch functions algebraically, for example by change  $x$  to  $x - x_1$  when a function has to be moved  $x_1$  steps to the right.
- Knowledge of suitable parametric transformations of functions, such as when cooling against the room temperature  $T_0$  can be modeled by the function  $y = C \cdot e^{k \cdot t} + T_0$ .
- Digital competency using tools, for example, how to get GeoGebra to create points from your measured data and then fit a suitable function with the regressions tools. Or how to enter a differential equation with initial conditions into Wolfram Alpha.
- Competence to critically evaluate the mathematical modeling process, be open to changing the model, and have an understanding for the errors that can be carried by the model and transferred over to the responses from the model to the questions they were given at the start, for example, the way a water parabola is affected by an air stream and the fact that this could not be modeled sufficiently by a parabola.
- Competence to organize, follow, and document the mathematical modeling process from start to finish.
- Competence to select the most essential parts and ignore the rest. As Albert Einstein said: "Make it as simple as possible, but not simpler."

Clearly, this is not something that is easily developed over the course of a few weeks. Rather you have to embrace a *modeling mind set* where modeling is the natural starting point of many, if not all, modules, lessons and assignments. To develop all these modeling sub competencies requires that this mind set is be transferred to the students over time. As the student's experiences accumulate, it is your job as a teacher to forge these experiences together to a meaningful whole. If you hold this modeling mind set central you will find that competencies such as the ability to reason, communicate, and calculate will come naturally as consequences of working with modeling problems.

## ABOUT THE MODELING PROBLEMS

Many of the problems we have selected might be experienced as difficult. Some of the problems are suitable for different mathematics courses, depending on your student's experiences and at the level of the mathematics that is taught. Remember, the aim of each problem is that it be solved over a long time so that the students will be able to understand the problem completely. The discussions the teacher will have with the students and the discussions the students have with each other as they tackle solutions to problems are generally very rewarding in that they help develop a student's language and conceptual understanding.

We have supplied most of the problems with extended solutions that might be seen as a way to write up a mathematical solution. Some of the problems can be used to show students how to write up the solution for the next problem.

The problems were selected to impart a certain breadth to the mathematical contents as well as to the problem solving techniques using GeoGebra. It means that it could be a good idea for you to first work through the solutions to several problems so that you will know how to help your students most effectively.

As authors, we like to share good mathematical modeling activities with our readers. If you have a suggestion for a good activity, preferably also with a solution, or a new way to solve one of our problems, please send it to us. You will receive a link to an Internet address where all contributed material will be published. It is our hope that in this way we will get a large inventory of suitable mathematical modeling activities that we all can use.

## ABOUT GEOGEBRA

GeoGebra is a dynamical mathematical laboratory suitable for teaching or learning all kind of mathematics including mathematical modeling. GeoGebra runs on all platforms and can be downloaded from [www.geogebra.org](http://www.geogebra.org).

GeoGebra is specifically built to show many different representations of dynamic objects at once. It handles everything you are likely to meet in secondary mathematics, and then more. While it may stop short of professional programs like MS Excel, Mathematica, and MATLAB that do pure number crunching, GeoGebra combines every feature you could wish for in a package free for educational use.

Some procedures are done more often than others. You will often need to enter measured values into GeoGebra's spreadsheet and create points from these values. You can zoom in and change labels on axes and change the background color. You can even change the font sizes and the number of decimals. We have chosen to describe these procedures in the running text as well as in Appendix A, *Introduction to GeoGebra*. In this way we hope that it not only will be possible to read the book from beginning to end but also to access a collection of modeling activities that you can dip into at any point.

We have used the Windows version of GeoGebra 5.0, which was released in 2014 but constantly updates with new functions, as described in the *change log* found at [http://wiki.geogebra.org/en/Reference:Changelog\\_5.0](http://wiki.geogebra.org/en/Reference:Changelog_5.0).

## ABOUT WOLFRAM ALPHA

Wolfram Alpha is a powerful mathematical Internet based tool that combines a search engine and a computer algebra system. It can solve equations, differential equations, and sketch graphs, and further find all sorts of mathematical facts as well as information on physics, chemistry, biology, and more. It complements GeoGebra very well, and it is seldom that a mathematics teacher would need any other tools than these two except for personal preferences. The best way to get familiar with Wolfram Alpha is to have a look at some of the many examples at <http://www.wolframalpha.com/examples/>.

## ABOUT TYPOGRAPHY

We have used the following conventions:

- Variables are written in *italic*:

$$x, y, p(x), t \dots$$

- Parameters are written in normal text. We do this so that you can separate independent and dependent *variables* from the parameters we can control, change, and fit to data:


$$y = k \cdot x + m, m(x) = a/x + 1, f(x) = a \cdot x^2 + b \cdot x + c \dots$$

- Input commands in GeoGebra's input bar, spread sheet, or Wolfram Alpha are written in **Courier bold**:

**=A2/B2, Fit[list1,m], Max = Extremum[f]...**

- Keyboard shortcuts, menu alternatives, and commands that are discussed rather than typed are written in **bold**:

**Ctrl-Shift-2, View – Spreadsheet...**, the **Locus** command

- Tool names are written in normal text, but often followed by a tool image: Use the Point tool  to create a point.

## ABOUT THE DIGITAL MATERIAL FOR THE BOOK

If you have the paper version of this book, it is in black and white. Any color references in the book refer to the GeoGebra file that we have created for the solutions to the problems. Download the GeoGebra solutions to all problems from the book's website. There is a GeoGebra file for just about every figure in the book, numbered accordingly.

You are also welcome to send similar problems with solutions directly to the authors so that you can access the materials other teachers have submitted. In this way we hope to enrich the book by creating new useful modeling teaching materials in an online resource bank.

## ABOUT THE AUTHORS

**Jonas Hall** is head of mathematics at Rodengymnasiet in Norrtälje, north of Stockholm, in Sweden, where he teaches mathematics and physics. He is a hobby mathematician with a special interest in problem solving, the aesthetics of mathematics, and teaching with technology. He is a multiple finalist in *Kappa*, a mathematical competition for mathematics teachers in Sweden held by the University of Stockholm, and he has dabbled in creating mathematical art, which can be found at <http://www.geogebrainstitut.se/art/art.asp>.

**Thomas Lingefjärd** is a professor of mathematics education at the University of Gothenburg. He started his teaching, however, as an upper secondary teacher in mathematics, physics, and ICT. Lingefjärd has for many years been interested in how the presence of modern technology can help students to learn and understand mathematics better. His main activities at the University of Gothenburg are to supervise PhD students, to teach courses in the mathematics teacher program, and to do research on the learning of mathematics.

Together, the authors are responsible for the Swedish GeoGebra institute, <http://www.geogebrainstitut.se>, one of over 100 regional and national GeoGebra institute over the world: <http://www.geogebra.org/institutes>. The aim of the institute is to spread the use of GeoGebra in Sweden. The authors have translated GeoGebra to Swedish and also written articles for teacher journals about GeoGebra. They also contribute to the international research around GeoGebra and its influence on mathematical teaching and learning in the Nordic–Baltic GeoGebra Network, <http://nordic.geogebra.no/> and run the Swedish Facebook support group.

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